The Mehler formula and the Green function of the multi-dimensional isotropic harmonic oscillator

1976 J. Phys. A: Math. Gen. 9 683

(http://iopscience.iop.org/0305-4470/9/5/004)

J. Phys. A: Math. Gen., Vol. 9, No. 5, 1976. Printed in Great Britain. © 1976

The Mehler formula and the Green function of the multidimensional isotropic harmonic oscillator†

J Bellandi Fo and E S Caetano Neto
Instituto de Física Teórica, Rua Pamplona 145, São Paulo, Brasil

Received 12 November 1975, in final form 5 January 1976

Abstract. Using a generalized Mehler formula, a closed representation is obtained for the Green function of the stationary Schrödinger equation for a multidimensional isotropic harmonic oscillator.

1. Derivation of the Green function

The present note contains a simple derivation of the Green function of the N dimensional harmonic oscillator using the generating function of the product of the harmonic oscillator wavefunction given by a generalized Mehler formula (Erdélyi 1953).

We define the Green function of the stationary Schrödinger equation for an N dimensional harmonic oscillator by means of the spectral decomposition

$$G^N(\boldsymbol{r}, \boldsymbol{r}', \lambda) = \sum_{\nu=0}^{\infty} (E_\nu - \lambda)^{-1} \sum_{\nu_1+\nu_2+\ldots+\nu_N=\nu} \psi_{\nu_1}(x_1)\psi_{\nu_1}(x_1') \ldots \psi_{\nu_N}(x_N)\psi_{\nu_N}(x_N') \tag{1}$$

where $\psi_{\nu_i}(x_i)$ is the harmonic oscillator wavefunction; $E_\nu = \nu + \frac{1}{2}N$ is the energy and $\boldsymbol{r}$ is a radius vector in the N dimensional space. (We put $\hbar = m = \omega = 1$.)

The generalized Mehler formula for the N dimensional harmonic oscillator can be written in the following way (see appendix)

$$\pi^{N/2} \sum_{\nu=0}^{\infty} \xi^\nu \sum_{\nu_1+\nu_2+\ldots+\nu_N=\nu} \psi_{\nu_1}(x_1)\psi_{\nu_1}(x_1') \ldots \psi_{\nu_N}(x_N)\psi_{\nu_N}(x_N')$$
$$= (1-\xi^2)^{-N/2} \exp[-\tfrac{1}{2}(\boldsymbol{r}^2 + \boldsymbol{r}'^2)] \exp\left(\frac{2\boldsymbol{r}\cdot\boldsymbol{r}'\xi - (\boldsymbol{r}^2+\boldsymbol{r}'^2)\xi^2}{1-\xi^2}\right). \tag{2}$$

We see that the Mehler formula contains explicity the full SU_N symmetry of the N dimensional harmonic oscillator.

We consider first the density matrix

$$\rho_\nu^N(\boldsymbol{r}, \boldsymbol{r}') = \operatorname{res} G(\boldsymbol{r}, \boldsymbol{r}', E_\nu) = \sum_{\nu_1+\nu_2+\ldots+\nu_N=\nu} \psi_{\nu_1}(x_1)\psi_{\nu_1}(x_1') \ldots \psi_{\nu_N}(x_N)\psi_{\nu_N}(x_N'). \tag{3}$$

† Work supported by Finep (Financiadora de Estudos e Projetos).

Using equation (2) we have

$$\rho_\nu^N(\boldsymbol{r},\boldsymbol{r}')=\pi^{-N/2}\exp[-\tfrac{1}{2}(\boldsymbol{r}^2+\boldsymbol{r}'^2)]\frac{1}{2\pi\mathrm{i}}\oint_{\xi=0}\mathrm{d}\xi\,\xi^{-(E_\nu-\frac{1}{2}N+1)}(1-\xi^2)^{-N/2}\times\exp\left(\frac{2\boldsymbol{r}\,.\,\boldsymbol{r}'\xi-(\boldsymbol{r}^2+\boldsymbol{r}'^2)\xi^2}{1-\xi^2}\right). \tag{4}$$

The Green function is given by

$$G^N(\boldsymbol{r},\boldsymbol{r}',\lambda)=\sum_{\nu=0}^{\infty}\frac{\rho_\nu^N(\boldsymbol{r},\boldsymbol{r}')}{E_\nu-\lambda} \tag{5}$$

and for $\mathrm{Re}(\frac{1}{2}N-\lambda)>0$ can be cast into the form

$$G^N(\boldsymbol{r},\boldsymbol{r}',\lambda)=\pi^{-N/2}\exp[-\tfrac{1}{2}(\boldsymbol{r}^2+\boldsymbol{r}'^2)]\int_0^1\mathrm{d}\xi\xi^{(N/2).\lambda-1}\times(1-\xi^2)^{-N/2}\exp\left(\frac{2\boldsymbol{r}\,.\,\boldsymbol{r}'\xi-(\boldsymbol{r}^2+\boldsymbol{r}'^2)\xi^2}{1-\xi^2}\right). \tag{6}$$

This integral representation is exactly the expression of Berendt and Weimar (1972) obtained using the full SU_N symmetry of the N dimensional harmonic oscillator in a Fock space representation. The singularities exhibited by equation (6) are the well known singularities of the harmonic oscillator Green function: none for $N=1$, a logarithmic one in the variable $|\boldsymbol{r}-\boldsymbol{r}'|$ for $N=2$ and a pole of order $N-2$ for higher N in that variable. A very simple variable transformation can also reproduce the results obtained by Bakhrakh *et al* (1972).

Acknowledgment

The authors wish to thank Professor A H Zimerman for critical reading of the manuscript.

Appendix

To prove equation (2) we start with the bilinear generating function of one N dimensional Hermite polynomial (Erdélyi 1953)

$$\sum\frac{t_1^{\nu_1}\dots t_N^{\nu_N}}{\nu_1!\dots\nu_N!}H_\nu(\boldsymbol{r})H_\nu(\boldsymbol{r}')=(t_1\dots t_N)^{-1}(\Delta_1\Delta_2)^{-1/2}\exp[\tfrac{1}{4}\psi_1(C\boldsymbol{r}+C\boldsymbol{r}')-\tfrac{1}{4}\psi_2(C\boldsymbol{r}-C\boldsymbol{r}')] \tag{A.1}$$

with the following notation:

$$\phi(\boldsymbol{r})=\sum_{i,j=1}^{N}C_{ij}x_ix_j \tag{A.2}$$

where C will be a fixed positive definite symmetric square matrix of real elements;

$$\begin{aligned}\phi_1(\boldsymbol{r}) &= \sum_{j=1}^{N} (x_j^2/t_j) + \phi(\boldsymbol{r}) \\ \phi_2(\boldsymbol{r}) &= \sum_{j=1}^{N} (x_j^2/t_j) - \phi(\boldsymbol{r})\end{aligned} \tag{A.3}$$

Δ_k is the determinant of ϕ_k, $k = 1, 2$. $\psi_k(\boldsymbol{r})$ is the reciprocal quadratic form of $\phi_k(\boldsymbol{r})$. For sufficiently small positive t_i, ϕ_k are positive definite.

If we put $t_1 = t_2 = \ldots t_N = \xi/2$ and $C_{ij} = 2\delta_{ij}$ we have for ϕ_k and ψ_k

$$\begin{aligned}\phi_1(\boldsymbol{r}) &= \sum_{i,j=1}^{N} (D_1)_{ij} x_i x_j; \qquad & \psi_1(\boldsymbol{r}) &= \sum_{i,j=1}^{N} (D_1^{-1})_{ij} x_i x_j \\ \phi_2(\boldsymbol{r}) &= \sum_{i,j=1}^{N} (D_2)_{ij} x_i x_j; \qquad & \psi_2(\boldsymbol{r}) &= \sum_{i,j=1}^{N} (D_2^{-1})_{ij} x_i x_j\end{aligned} \tag{A.4}$$

where $(D_{1,2})_{ij} = 2(1 \pm \xi)\delta_{ij}/\xi$, and equation (A.1) can be transformed into

$$\sum \frac{(\xi/2)^{\nu}}{\nu_1! \ldots \nu_N!} H_\nu(\boldsymbol{r}) H_\nu(\boldsymbol{r}') = (1-\xi^2)^{-N/2} \exp\left(\frac{2\boldsymbol{r} \cdot \boldsymbol{r}'\xi - (\boldsymbol{r}^2 + \boldsymbol{r}'^2)\xi^2}{1-\xi^2}\right). \tag{A.5}$$

References

Bakhrakh V L, Vetchinkin S I and Khristenko S V 1972 *Teor. Mat. Fiz.* **12** 776

Berendt G and Weimar L 1972 *Lett. Nuovo Cim.* **5** 613

Erdélyi A (ed) 1953 *Higher Transcendental Functions* vol 2 (New York: McGraw-Hill) p 287